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Geometrically non-linear equations in the theory of momentless shells with applications to problems on the non-classical forms of loss of stability of a cylinder☆

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Abstract

Geometrically non-linear and linearized equations in the theory of momentless shells are set up based on the kinematic relations in [Paimushin VN, Shalashilin VI. Relations of the theory of deformations in the quadratic approximation and problems of constructing refined versions of the geometrically non-linear theory of laminar structural components. *Prikl Mat Mekh* 2005; **69**(5): 861–81]. The use of these equations, unlike in the case of the well-known equations, enables one to avoid the occurrence of spurious bifurcation points in solving real problems. Non-classical problems of the stability of cylindrical shells under an external pressure, axial compression and torsion are considered, which can be formulated on the basis of the derived equations of the theory of momentless shells. Their exact analytical solutions are found and enable one to estimate the quality of the previously obtained relations [Paimushin VN, Shalashilin VI. Relations of the theory of deformations in the quadratic approximation and problems of constructing refined versions of the geometrically non-linear theory of laminar structural components. *Prikl Mat Mekh* 2005; **69**(5): 861–81] and the richness of content of the equations which have been constructed compared with well-known equations in the mechanics of thin shells. It is established that the majority of the new forms of loss of stability of cylindrical shells which are revealed relate to a number of shear forms, the onset of which is possible before the flexural forms which have been well studied up to now, in the case of small values of the shear modulus of a shell material with a very highly pronounced anisotropy in its properties.
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1. Consistent geometrically non-linear and linearized equations of the theory of momentless shells

Assuming that the middle surface of a shell σ is referred to its lines of principal curvature x^1 and x^2, we will introduce the following notation: A_1 and A_2 are Lamé parameters, $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{m}$ are the unit vectors of the tangents and normal to σ, k_1 and k_2 are the curvatures of the coordinate lines and $\mathbf{u} = u_1\mathbf{e}_1 + u_2\mathbf{e}_2 + w\mathbf{m}$ is the vector of the displacements of points of σ.

In this notation the tensile deformations ε_i and the shear deformation $\sin\gamma_{12}$ are calculated in a consistent quadratic approximation[1] using the formulae

$$\varepsilon_1 = e_{11} + (e_{12}^2 + \omega_1^2)/2, \quad \overset{\longrightarrow}{\underset{\longleftarrow}{1, 2}} \tag{1.1}$$

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$$\sin\gamma_{12} = 2\varepsilon_{12} = (1+e_{11})e_{21} + (1+e_{22})e_{12} + \omega_1\omega_2 \qquad (1.2)$$

where

$$e_{11} = \frac{1}{A_1}u_{1,1} + \frac{A_{1,2}}{A_1A_2}u_2 + k_1 w$$
$$e_{12} = \frac{1}{A_1}u_{2,1} - \frac{A_{1,2}}{A_1A_2}u_1, \quad \omega_1 = \frac{1}{A_1}w_{,1} - k_1u_1, \quad \overrightarrow{\underleftarrow{1,2}} \qquad (1.3)$$

The following expression for the variation in the potential energy of deformation corresponds to the above kinematic relations

$$\delta U = \iint_\sigma (T_{11}\delta\varepsilon_1 + 2T_{12}\delta\varepsilon_{12} + T_{22}\delta\varepsilon_2)A_1A_2dx^1dx^2 =$$
$$= \iint_\sigma [S_{11}\delta e_{11} + S_{12}\delta e_{12} + S_{21}\delta e_{21} + S_{22}\delta e_{22} + S_{13}\delta\omega_1 + S_{23}\delta\omega_2]A_1A_2dx^1dx^2$$

where ($T_{21} \equiv T_{12}$)

$$S_{11} = T_{11} + T_{12}e_{21}, \quad S_{12} = T_{11}e_{12} + T_{12}(1+e_{22}), \quad \overrightarrow{\underleftarrow{1,2}} \qquad (1.4)$$

$$S_{13} = T_{11}\omega_1 + T_{12}\omega_2, \ \overrightarrow{\underleftarrow{1,2}} \qquad (1.5)$$

We will assume that the edge of the shell coincides with the coordinate lines $x^i = x^i_-, x^i_+ (i = 1, 2)$, the boundary forces

$$\mathbf{P}_1 = p_{11}\mathbf{e}_1 + p_{12}\mathbf{e}_2, \quad \mathbf{P}_2 = p_{21}\mathbf{e}_1 + p_{22}\mathbf{e}_2 \qquad (1.6)$$

are specified on them and that the surface forces, specified by the vector

$$\mathbf{P} = p_1\mathbf{e}_1 + p_2\mathbf{e}_2 + p_3\mathbf{m} \qquad (1.7)$$

are applied to points of the surface σ. According to the principle of virtual displacements, a variation in the work done by these forces and the variation δU must satisfy the equation $\delta U - \delta A = 0$, which, after the traditional transformations, reduces to the form

$$\sum_{i=1}^{2}\int_{x_-^{3-i}}^{x_+^{3-i}} [(S_{i1} - p_{i1})\delta u_1 + (S_{i2} - p_{i2})\delta u_2 + S_{i3}\delta w]A_{3-i}dx^{3-i} -$$
$$-\iint_\sigma (f_1\delta u_1 + f_2\delta u_2 + f_3\delta w)dx^1dx^2 = 0 \qquad (1.8)$$

The equilibrium equations

$$f_1 = (A_2S_{11})_{,1} + (A_1S_{21})_{,2} - A_{2,1}S_{22} + A_{1,2}S_{12} + A_1A_2(k_1S_{13} + p_1) = 0, \ \overrightarrow{\underleftarrow{1,2}}$$
$$f_3 = (A_2S_{13})_{,1} + (A_1S_{23})_{,2} - A_1A_2(k_1S_{11} + k_2S_{22} - p_3) = 0 \qquad (1.9)$$

for which the boundary conditions are formulated from the conditions that the contour integrals in Eq. (1.8) are equal to zero, follow from this.

It should be noted that all these equations only differ from the well-known equations (see Ref. 2, for example) in the expressions for S_{11} and S_{22} (1.4), which are consistent with the kinematic relations (1.1) and (1.2).

If the shell material is linearly elastic, then the forces T_{11}, T_{22}, T_{12} appearing in relations (1.4) and (1.5) are associated with the deformations (1.1) and (1.2) by elasticity relations of the form (using the generally accepted notation)

$$T_{11} = B_{11}(\varepsilon_1 + \nu_{21}\varepsilon_2), \quad T_{22} = B_{22}(\varepsilon_2 + \nu_{12}\varepsilon_1), \quad T_{12} = 2B_{12}\varepsilon_{12} \qquad (1.10)$$

where $B_{11} = E_1 t/(1-\nu_{12}\nu_{21})$, $B_{22} = E_2 t/(1-\nu_{12}\nu_{21})$ and $B_{12} = G_{12}t$ are the corresponding stiffnesses and t is the thickness of the shell.

We will now consider two equilibrium states of a shell. Suppose the first of them, the unperturbed state, is characterized by the forces T_{11}^0, T_{22}^0, T_{12}^0. On introducing the standard assumption that, in the first state, the shell is stressed but not deformed, linearizing the non-linear equations which have been setup in the neighbourhood of the unperturbed state and retaining the earlier notation for the parameters of the stress-strain state (SSS) of the perturbed state, the following linearized equations for the perturbed state can be set up

$$\begin{aligned} f_1 &= (A_2S_{11})_{,1} + (A_1S_{21})_{,2} - A_{2,1}S_{22} + A_{1,2}S_{12} + A_1A_2k_1S_{13} = 0, \quad \overrightarrow{\underleftarrow{1,2}} \\ f_3 &= (A_2S_{13})_{,1} + (A_1S_{23})_{,2} - A_1A_2(k_1S_{11} + k_2S_{22}) = 0 \end{aligned} \tag{1.11}$$

in which

$$S_{11} = T_{11} + T_{12}^0 e_{21}, \quad S_{12} = T_{11}^0 e_{12} + T_{12}^0 e_{22} + T_{12}, \quad S_{13} = T_{11}^0\omega_1 + T_{12}^0\omega_2, \quad \overrightarrow{\underleftarrow{1,2}} \tag{1.12}$$

and, unlike relations (1.10),

$$T_{11} = B_{11}(e_{11} + \nu_{21}e_{22}), \quad T_{22} = B_{22}(e_{22} + \nu_{12}e_{11}), \quad T_{12} = B_{12}(e_{12} + e_{21}) \tag{1.13}$$

2. Shear and other non-classical forms of loss of stability of a circular cylindrical shell acted upon by external pressure, axial compression and torsion

In the case of a cylindrical shell of radius R referred to axial coordinates ($x^1 = x$) and peripheral coordinates ($x^2 = \theta$), we have

$$A_1 = 1, \quad A_2 = R, \quad k_1 = 0, \quad k_2 = 1/R$$

In the case of the above-mentioned forms of loading, the linear equations of the momentless theory yield the solutions

$$T_\theta^0 = T_{22}^0 = -pR, \quad T_x^0 = T_{11}^0 = -p_{11}, \quad T_{x\theta}^0 = T_{12}^0 = p_{12} \tag{2.1}$$

where p is the external pressure and p_{11} and p_{12} are the linear axial compressive and torsional forces on the ends of the shell.

In the case being considered, formulae (1.3), on introducing the notation $u_1 = u$, $u^2 = v$, take the form

$$e_{11} = u_{,x}, \quad e_{22} = \frac{v_{,\theta} + w}{R}, \quad e_{12} = v_{,x}, \quad e_{21} = \frac{u_{,\theta}}{R}, \quad \omega_1 = w_{,x}, \quad \omega_2 = \frac{w_{,\theta} - v}{R} \tag{2.2}$$

and, substituting into relations (1.13), we obtain

$$\begin{aligned} T_x &= T_{11} = B_{11}\left(u_{,x} + \nu_{21}\frac{v_{,\theta} + w}{R}\right), \quad T_\theta = T_{22} = B_{22}\left(\frac{v_{,\theta} + w}{R} + \nu_{12}u_{,x}\right) \\ T_{x\theta} &= T_{12} = B_{12}\left(v_{,x} + \frac{u_{,\theta}}{R}\right) \end{aligned} \tag{2.3}$$

In accordance with relations (2.1), we now consider the three forms of distinct loading of a shell.

External pressure ($T_{11}^0 = T_{12}^0 = 0$, $T_{22}^0 = T_\theta^0 = -pR$). Since, in our case,

$$S_{11} = T_x, \quad S_{22} = T_\theta, \quad S_{12} = T_{x\theta}, \quad S_{21} = T_{x\theta} - pRe_{21}, \quad S_{13} = 0, \quad S_{23} = -pR\omega_2 \tag{2.4}$$

Eqs. (1.11) then take the form

$$RT_{x,x} + T_{x\theta,\theta} - pu_{,\theta\theta} = 0, \quad T_{\theta,\theta} + RT_{x\theta,x} - p(w_{,\theta} - v) = 0, \quad T_\theta = 0 \tag{2.5}$$

Eliminating $T_{x\theta}$ from the first equation, we obtain

$$R^2T_{x,xx} - Rpu_{,x\theta\theta} + p(w_{,\theta} - v)_{,\theta} = 0 \tag{2.6}$$

When account is taken of the equality $T_\theta = 0$, it follows from the physical relations (2.3) that

$$T_x = B_1 u_{,x}, \quad B_1 = B_{11}(1 - \nu_{21}\nu_{12}), \quad T_{x\theta} = B_{12}(\upsilon_{,x} + u_{,\theta}/R) \tag{2.7}$$

We will first consider the case when the shell retains an axi-symmetric form in the perturbed state, that is, when the unknowns in Eqs. (2.5) and (2.7) are solely functions of x. Relations (2.5) and (2.7) then reduce to the following two equations (a prime denotes differentiation with respect to x)

$$u'' = 0, \quad RB_{12}\upsilon'' + p\upsilon = 0 \tag{2.8}$$

The second equation has the general solution

$$\upsilon = c_1 \sin kx + c_2 \cos kx, \quad k^2 = pR/B_{12}$$

The occurrence of adjacent forms of equilibrium when $\upsilon \neq 0$, i.e. bifurcation of the solution becomes possible when

$$p = p_*^{(1)} = \pi^2 (R/L^2) B_{12} = \pi^2 t (R/L^2) G_{12} \tag{2.9}$$

At the same time, different forms of loss of stability (FLS) are possible depending on the condition for the clamping of the edges. If the edges are clamped such that peripheral displacements are eliminated $\upsilon(0) = \upsilon(L) = 0$, then

$$\upsilon = c_1 \sin(\pi x/L) \tag{2.10}$$

In this case, the middle section of the shell ($x = L/2$) is rotated with respect to the ends through an angle c_1/R. However, in the case of ends which are fixed in the peripheral direction ($\upsilon'(0) = \upsilon'(L) = 0$)

$$\upsilon = c_2 \cos(\pi x/L) \tag{2.11}$$

In the case of this form of loss of stability, one of the ends rotates with respect to the other through an angle $2c_2/R$.

The above form of loss of stability of a shell is explained by the special features of the behaviour of the load, which preserves its direction during perturbations, that is, it ceases to be normal to the shell surface. Loads of this form are shown in Fig. 1 by the solid arrows in the unperturbed state and by the dashed arrows in the perturbed state, which corresponds to a peripheral displacement υ. In the case of such a load, a peripheral component $p\upsilon/R$ appears in the perturbed state. An element of a shell $dx \cdot Rd\theta$ is shown in Fig. 2, and it follows from the condition for the equilibrium of this element in projections onto the peripheral direction that

$$dT_{x\theta} + p(\upsilon/R)dx = 0$$

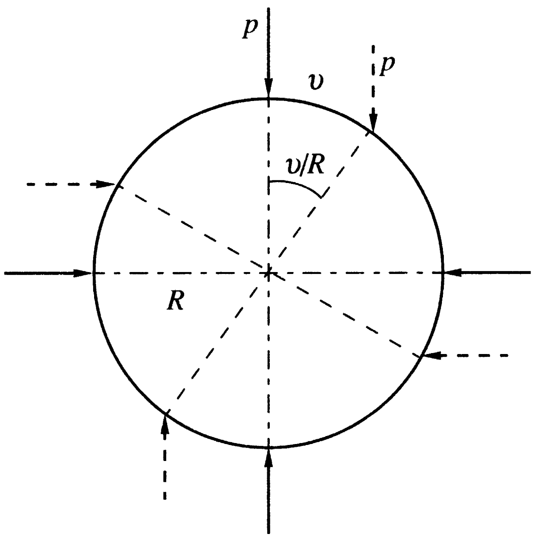

Fig. 1.

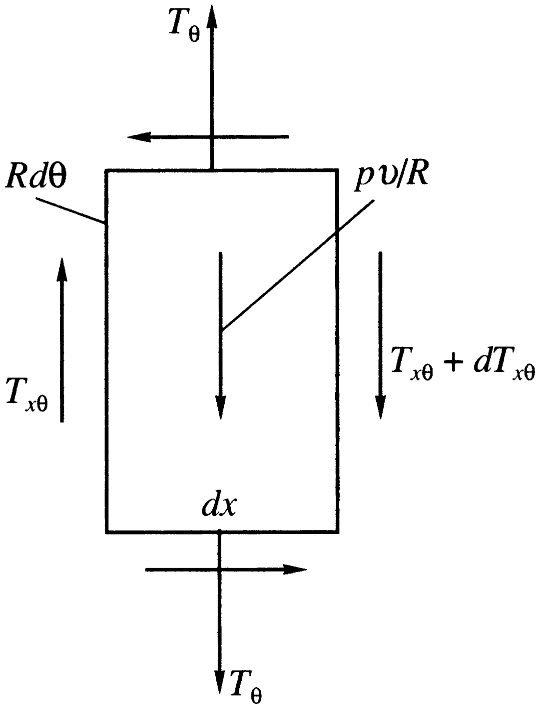

Fig. 2.

It can be seen from relations (2.7) that $T_{x\theta} = B_{12}v'$ in the case of this deformation. As a result, we arrive at the second equation of (2.8)

We will now investigate the case when

$$u = U(x)\cos\theta, \quad v = V(x)\sin\theta, \quad w = W(x)\cos\theta \tag{2.12}$$

We shall initially consider the case when U, V and W are constant with respect to x. It then follows from the second and third equations of (2.5) that

$$W = -V$$

From relations (2.7) we obtain that $T_{x\theta} = B_{12}u$, θ/R. Hence, the first equation of (2.5) reduces to the equality

$$(B_{12}/R - p)u_{,\theta\theta} = 0$$

The expression for the critical pressure

$$p_*^{(2)} = B_{12}/R = G_{12}t/R \tag{2.13}$$

follows from this.

The corresponding form of loss of stability when the ends of the shell are clamped such that $W = V = 0$ is shown in Fig. 3a.

In the general case, when U, V and W are variable, we obtain from relations (2.7) that

$$T_{x\theta} = B_{12}(V' + U/R), \quad T_x = B_1U'$$

Since $T_\theta = 0$, it then follows from expressions (2.3) and (2.12) that

$$W + V = -R\nu_{12}U' \tag{2.14}$$

Equations (2.5) therefore reduce to the form

$$B_1RU''' + p(1 + \nu_{12})U' = 0, \quad V'' + (1/R - p\nu_{12}/B_{12})U' = 0 \tag{2.15}$$

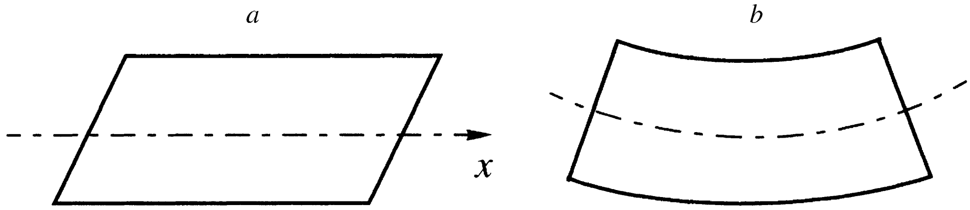

Fig. 3.

Subject to the condition that

$$U(L/2) = 0, \quad U'(0) = U(L) = 0 \tag{2.16}$$

the first equation of (2.15) has the non-trivial solution

$$U = C\sin(\pi x/L) \tag{2.17}$$

when

$$p = p_*^{(3)} = \pi^2 B_1 R/[(1+\nu_{12})L^2] \tag{2.18}$$

In this case, it follows from equality (2.14) that

$$W + V = -C\nu_{12}\pi(R/L)\cos(\pi x/L) \tag{2.19}$$

The corresponding form of loss of stability is shown in Fig. 3b. A similar form of loss of stability was discovered previously[3] when analysing the stability of a strip.

The second equation of (2.15) also has a non-trivial solution $U \neq 0$ for $V = 0$ when

$$p = p_*^{(4)} = \nu_{12}B_{12}/R \tag{2.20}$$

But, since $\nu_{12} < 1$, then $p_*^{(4)} < p_*^{(2)}$ and this form of loss of stability cannot be realized.

Uniform axial compression ($T_{11}^0 = T_x^0 = -q$, $T_{22}^0 = T_{12}^0 = 0$). In this case, Eq. (1.11) take the form

$$RT_{x,x} + T_{x\theta,\theta} = 0, \quad T_{\theta,\theta} + RT_{x\theta,x} - Rqv_{,xx} = 0, \quad T_\theta + Rqw_{,xx} = 0 \tag{2.21}$$

We shall confine ourselves to considering the cases when the functions have zero variability in the peripheral direction, that is, when

$$u = u(x), \quad v = v(x), \quad w = w(x) \tag{2.22}$$

Together with the last relation of (2.3), the second equation of (2.21) then takes the form

$$T'_{x\theta} - qv'' = 0, \quad T_{x\theta} = B_{12}v'$$

and, from this, we obtain the equation

$$(B_{12} - q)v'' = 0$$

which has a non-trivial solution when

$$q = q_*^{(1)} = B_{12}, \quad \text{or} \quad \sigma_x^* = G_{12} \tag{2.23}$$

In the case of such a form of loss of stability, if one of the ends of the shell is fixed, such that peripheral displacements are eliminated, then the other end rotates relative to it while remaining parallel to it. This form of loss of stability is analogous to the shear loss of stability of a strip under uniform transverse compression only, in this case, the strip would be wrapped into a ring.

The first and last equations of (2.21) reduce to the form

$$T'_x = 0, \quad T_\theta + Rqw'' = 0 \tag{2.24}$$

When account is taken of the equilibrium of the shell in projections onto its axis, it follows from the first equation of (2.24) that $T_x = 0$. It therefore follows from the first relation of (2.3) that $u_{,x} = -\nu_{21} w/R$. Then, from the second relation of (2.3), we obtain

$$T_\theta = B_2 w/R; \quad B_2 = B_{22}(1 - \nu_{12}\nu_{21})$$

and the second equation of (2.24) takes the form

$$R^2 q w_{,xx} + B_2 w = 0 \tag{2.25}$$

This equation shows that related forms of equilibrium of the type

$$w = c\sin(\pi x/L) \tag{2.26}$$

appear in the case of a shell when

$$q = q_*^{(2)} = B_2 L^2/(\pi^2 R^2) \tag{2.27}$$

This result is easy to understand if one considers the stability of a momentless bar on an elastic Winkler base, an element of which in the perturbed state is shown in Fig. 4a, and the peripheral forces play the role of the forces of interaction between the bar and the base. It follows from the condition for the element to be in equilibrium that

$$Pw'' + \alpha w = 0$$

Here α is the coefficient of the bed of the base. Related forms of equilibrium (2.26) occur when

$$P = P_* = \alpha L^2/\pi^2 \tag{2.28}$$

If we consider an elementary longitudinal strip of a momentless shell of width $Rd\theta$, the cross-section of which is shown in Fig. 4b, as a rod, then $P = qRd\theta$ and, consequently,

$$\alpha w = T_\theta d\theta = B_2(w/R)d\theta$$

After substituting α and P from these relations into equality (2.28), we arrive at formula (2.27).

Pure torsion of a shell ($T_{11}^0 = T_{22}^0 = 0$, $T_{12}^0 = s$). In this case where there is zero variability in the peripheral direction, that is, when the representations (2.22) hold, Eqs. (1.11) take the form

$$RT'_x = 0, \quad RT'_{x\theta} + 2sw' = 0, \quad T_\theta + 2sv' = 0 \tag{2.29}$$

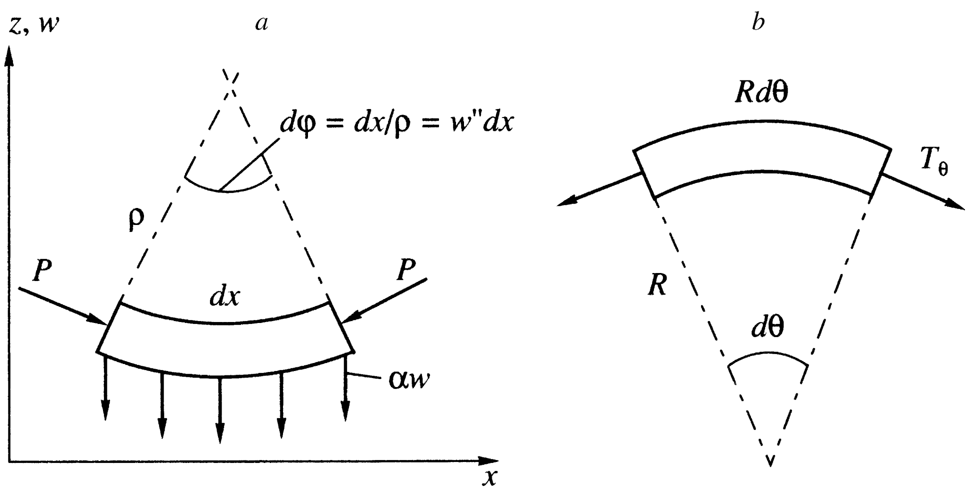

Fig. 4.

It follows from the first equation of (2.29) and the condition for the shell to be in equilibrium in projections onto the x-axis that $T_x = 0$. Hence, from relations (2.3), we obtain

$$u' + \nu_{21} w/R = 0 \tag{2.30}$$

When account is taken of relations (2.3), the second and third equations of (2.29) reduce to the form

$$B_{12} R \upsilon'' + 2sw' = 0, \quad B_{22}(w/R + \nu_{12} u') + 2s\upsilon' = 0 \tag{2.31}$$

On eliminating u and w from these equations using equality (2.30), we arrive at the equation

$$(B_2 B_{12} - 4s^2)\upsilon'' = 0$$

The non-trivial solution of this equation when $\upsilon(0) = 0$ has the form

$$\upsilon = Cx$$

and is possible when

$$s = s_* = \sqrt{B_2/B_{12}}/2 \tag{2.32}$$

At the same time, as can be seen from equality (2.30) and the first equation of (2.31), the displacements u and w are given by the relations

$$u = \nu_{21}(w/R)cx, \quad w = -c\sqrt{B_{12}/B_2} \tag{2.33}$$

if $u(0) = 0$.

It can be seen from these relations that the form of loss of stability accompanying torsion is a pure shear form.

3. Simplified versions of the geometrically non-linear theory of momentless shells and an analysis of the possibility of describing pure shear forms of loss of stability in the case of different versions of loading

3.1. Non-linear theory of mean flexure

The so-called mean flexure theory of shells[3] is the most frequently used version of the non-linear theory of thin shells in practical applications. Within the framework of this theory the shear deformations of the middle surface are determined using the formulae

$$\varepsilon_1 = e_{11} + \omega_1^2/2, \quad \varepsilon_2 = e_{22} + \omega_2^2/2, \quad 2\varepsilon_{12} = e_{12} + e_{21} + \omega_1\omega_2 \tag{3.1}$$

and expressions for the flexural deformations of shells are used in the geometrically linear approximation.

When expressions (3.1) are used instead of relations (1.1) and (1.2), we can write equalities (1.4) in the form

$$S_{11} = T_{11}, \quad S_{22} = T_{22}, \quad S_{12} = S_{21} = T_{12} \tag{3.2}$$

while formulae (1.5) remain unchanged.

The equilibrium equations (1.9) are simplified by virtue of equalities (3.2). From them we obtain the linearized equations of the perturbed state

$$\begin{gathered} f_1 = (A_2 T_{11})_{,1} + (A_1 T_{12})_{,2} - A_{2,1} T_{22} + A_{1,2} T_{12} + A_1 A_2 k_1 S_{13} = 0, \quad \overrightarrow{\underleftarrow{1,2}} \\ f_3 = (A_2 S_{13})_{,1} + (A_1 S_{23})_{,2} - A_1 A_2 k_1 T_{11} + k_2 T_{22} = 0 \end{gathered} \tag{3.3}$$

in which the forces are determined using formulae (1.12) for S_{i3} and (1.13) for T_{ij}.

With the aim of investigating the quality of the equations derived, we will now analyse the problems considered in Section 2 which were formulated on the basis of these equations, starting out from the most general equations.

3.2. External pressure

By virtue of equalities (3.2), supplemented with the equalities

$$S_{13} = 0, \quad S_{23} = T_{22}^0\omega_2 = -pR\omega_2$$

instead of Eqs. (2.5), we shall have the simplified equations

$$f_1 = L_1(u, \upsilon, w) = 0, \quad f_2 = L_2(u, \upsilon, w) - \tilde{p}R\left(\frac{\partial w}{\partial\theta} - \upsilon\right) = 0$$
$$f_3 = L_3(u, \upsilon, w) - \tilde{p}R\left(\frac{\partial^2 w}{\partial\theta^2} - \frac{\partial\upsilon}{\partial\theta}\right) = 0; \quad \tilde{p} = \frac{p}{B_{22}} \tag{3.4}$$

where

$$L_1(u, \upsilon, w) = \frac{\partial^2 u}{\partial\eta^2} + g_1\frac{\partial^2 u}{\partial\theta^2} + (\nu_{21} + g_1)\frac{\partial^2\upsilon}{\partial\eta\partial\theta} + \nu_{21}\frac{\partial w}{\partial\eta}$$
$$L_2(u, \upsilon, w) = \frac{\partial^2\upsilon}{\partial\theta^2} + g_2\frac{\partial^2\upsilon}{\partial\eta^2} + (\nu_{12} + g_2)\frac{\partial^2 u}{\partial\eta\partial\theta} + \frac{\partial w}{\partial\theta}$$
$$L_3(u, \upsilon, w) = -\left(\frac{\partial\upsilon}{\partial\theta} + w + \nu_{12}\frac{\partial u}{\partial\eta}\right); \quad g_1 = \frac{B_{12}}{B_{11}}, \quad g_2 = \frac{B_{12}}{B_{22}}$$

When $\partial/\partial\theta = 0$, it is easy to show that the bifurcation value of the load (2.9) and the shear forms of loss of stability (2.10) and (2.11) are determined by the solutions of Eqs. (3.4).

If the functions appearing in Eqs. (3.4) are represented in the form

$$u = U\sin n\theta, \quad \upsilon = V\cos n\theta, \quad w = W\sin n\theta \tag{3.5}$$

we arrive at the system of equations

$$U'' - g_1U - (\nu_{21} + g_1)V' + \nu_{21}W' = 0$$
$$(1 - \tilde{p}R)(W - V) + g_2V'' + (\nu_{12} + g_2)U' = 0 \tag{3.6}$$
$$(1 - \tilde{p}R)(W - V) + \nu_{12}U' = 0$$

Just a single bifurcation value of the load

$$p_*^{(3)}R = B_{22} \tag{3.7}$$

follows from these equations when there is no variability of the functions in the axial direction whereas two bifurcation values, determined using formulae (2.13) and (3.7), follow from the unsimplified equations and, moreover, $p_*^{(2)} < p_*^{(3)}$. Furthermore, it should be noted that formula (2.13), which was obtained in a rigorous formulation of the problem, has a clear physical meaning, while the mechanism of the realization of loss of stability in the case of the functions $u_0 = 0$, $w_0 = -v_0 = \text{const}$, that is, when

$$u \equiv 0, \quad w = w_0\sin\theta, \quad \upsilon = -w_0\cos\theta \tag{3.8}$$

and the bifurcation value of the load according to formula (3.7) do not lend themselves, in general, to any physical explanation.

If we reject the assumption of zero variability of the functions U, V and W in the axial direction and represent the solution in the form

$$U = u_0\cos\lambda\eta, \quad V = \upsilon_0\sin\lambda\eta, \quad W = w_0\sin\lambda\eta; \quad \lambda = \pi R/L$$

we arrive at the equations

$$\begin{aligned}&-(\lambda^2+g_1)u_0-(\nu_{21}+g_1)\lambda\upsilon_0+\nu_{21}\lambda w_0 = 0\\&-(\nu_{12}+g_2)\lambda u_0-(1-mg_2+g_2\lambda^2)\upsilon_0+(1-mg_2)w_0 = 0\\&-\nu_{12}\lambda u_0-(1-mg_2)\upsilon_0+(1-mg_2)w_0 = 0;\quad m = p/R\end{aligned} \tag{3.9}$$

It can be shown that a non-trivial solution of system (3.9) exists in the case of a bifurcation value of the load p, which is determined using the formula

$$p_*^{(4)} = E_2t/R \tag{3.10}$$

This result differs fundamentally from the results which follow from the analogous unsimplified equations and does not lend itself to any physical explanation whatsoever.

3.3. Axial compression

Since equalities (3.2), supplemented by the equalities

$$S_{13} = T_{11}^0\omega_1,\quad S_{23} = 0$$

hold within the framework of the model being used, then, in the case under consideration, instead of (3.4) we arrive at the equations

$$\begin{aligned}&f_1 = L_1(u,\upsilon,w) = 0,\quad f_2 = L_2(u,\upsilon,w) = 0\\&f_3 = L_3(u,\upsilon,w)-\tilde{q}\frac{\partial^2 w}{\partial\eta^2} = 0;\quad \tilde{q} = \frac{q}{B_{22}}\end{aligned} \tag{3.11}$$

In the case of zero variability of the functions in the peripheral direction, Eq. (3.11) take the form

$$U''+\nu_{21}W' = 0,\quad g_2V'' = 0,\quad \tilde{q}W''+W+\nu_{12}U' = 0 \tag{3.12}$$

It follows from the first two equations of (3.12) that

$$V\equiv 0,\quad U' = -\nu_{21}W$$

Hence, the last equation of (3.12) is transformed to the form

$$W''+k^2W = 0,\quad k^2 = E_2t/q$$

It can be shown that the general solution of the last equation, when the condition $w(\eta = 0) = 0$ is imposed, yields the form of loss of stability

$$w = c_1\sin k\eta \tag{3.13}$$

and the bifurcation value of the load (2.27).

3.4. Pure torsion

Within the model used for the forces, we have formulae (3.2), supplemented with the equalities

$$S_{13} = s\omega_2,\quad S_{23} = s\omega_1$$

and, using these, we can write the stability equations (3.4) in the form

$$f_1 = L_1(u, \upsilon, w) = 0, \quad f_2 = L_2(u, \upsilon, w) + \tilde{s}\frac{\partial w}{\partial \eta} = 0$$
$$f_3 = L_3(u, \upsilon, w) + 2\tilde{s}\left(\frac{\partial^2 w}{\partial\eta\partial\theta} - \frac{\partial \upsilon}{\partial \eta}\right) = 0; \quad \tilde{s} = \frac{s}{B_{22}} \tag{3.14}$$

It can be shown that the solution of Eqs. (3.14) for the case of zero variability of the functions in the peripheral direction yields the bifurcation value of the load

$$s = s_* = t\sqrt{G_{12}E_2/2} \tag{3.15}$$

which is twice as large compared with the value (2.32).

3.5. Equations corresponding to the use of kinematic relations in the incomplete quadratic approximation

If the simplified relation

$$\gamma_{12} = e_{21} + e_{12} + \omega_1\omega_2 \tag{3.16}$$

is used to determine the shear deformation instead of (1.2), then, instead of formulae (1.4), we arrive at the formulae

$$S_{11} = T_{11}, \quad S_{12} = T_{12} + T_{11}e_{12}, \quad \overrightarrow{\underleftarrow{1, 2}} \tag{3.17}$$

for the forces in Eqs. (1.9), while formulae (1.5) remain unchanged.

By virtue of equalities (3.17), the forces occurring in the linearized equations of the perturbed state (1.1) will be determined using the formulae

$$S_{11} = T_{11}, \quad S_{12} = T^0_{11}e_{12} + T_{12}, \quad S_{13} = T^0_{11}\omega_1 + T^0_{12}\omega_2, \quad \overrightarrow{\underleftarrow{1, 2}} \tag{3.18}$$

An investigation of the above equations showed that, in the case of external pressure and axial compression, they lead to the same results as those indicated in Section 2. However, in the case of pure torsion, their solution leads, as above, to formulae (3.15) and a bifurcation value s_* which is twice as large as that found using formula (2.32).

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